Discrete Mathematics 60 (1986) 193–197
North-Holland

POINT SETS WITH MANY UNIT CIRCLES

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Received 25 January 1985

Every three of n points in the plane determine a circle. The maximum number $f(n)$ of congruent circles is determined for $n \leqslant 7$: $f(3)=1$, $f(4)=f(5)=4$, $f(6)=8$, $f(7)=12$.

1. Introduction

Every noncollinear set of three of n points in the plane determines a circle. What is the maximum number $f(n)$ of congruent, say unit circles? Erdös [2, 3] raised this problem, and he observed that

$$cn \leqslant f(n) \leqslant \tfrac{1}{3}n(n-1). \tag{1}$$

The left inequality is trivial. A proof of the right inequality could be as follows: Every pair of points lies on at most two unit circles, and then each circle is counted at least three times; hence $f(n) \leqslant \frac{2}{3}\binom{n}{2} = \frac{1}{3}n(n-1)$.

Elekes [1] gave a nice construction, which proves

$$f(n) > cn^{\frac{3}{2}} \tag{2}$$

as follows: Consider x unit circles which intersect one another in one point. These circles pairwise have $n=\binom{x}{2}$ simple points of intersection. Every three of them determine a unit circle (see Lemma 1), so that

$$f(n) = f\left(\binom{x}{2}\right) \geqslant \binom{x}{3} + x, \qquad x \geqslant 4, \tag{3}$$

which yields (2).

In this paper $f(n)$ is determined for $n = 5, 6, 7$.

Theorem. *If $f(n)$ is the largest integer for which there are n points in the plane so that there are $f(n)$ distinct unit circles which pass through at least three of the n points, then $f(3)=1$, $f(4)=f(5)=4$, $f(6)=8$, $f(7)=12$.*

2. Lemmas

Lemma 1. (a) *If three unit circles have one point of intersection in common, then the three remaining points of intersection determine another unit circle.*

(b) *Three unit circles determined by the pairs of three points on a given unit circle have one point in common.*

Proof. (a) Consider the three unit vectors from the common point of intersection of three unit circles to their centers. The three sums of pairs of these vectors determine the remaining points of intersection, and the sum of all three vectors determines the center of the desired unit circle.

(b) Consider the three unit vectors from the center of a unit circle to three points of this circle. The three sums of pairs of these vectors determine the centers of the three unit circles, and the sum of all three vectors is a common point of these three circles. □

Lemma 2. *Each of n points lies on at most $n-1$ unit circles which pass through at least three of these n points. If one point lies on exactly $n-1$ circles, then each circle contains exactly three of the n points, and all n points are intersections of the $n-1$ circles.*

Proof. Consider t unit circles which have one point in common, and the ith of which passes through $p_i \geqslant 3$ of n points, $1 \leqslant i \leqslant t$. If k_1 respectively k_2 denotes the number of points which lie on exactly 1, respectively 2, of these circles, then

$$k_1 + 2k_2 + t = \sum_{i=1}^{t} p_i. \tag{4}$$

This yields

$$t = k_1 + k_2 - \frac{1}{2}\left(k_1 + \sum_{i=1}^{t} (p_i - 3)\right) \leqslant k_1 + k_2 \leqslant n - 1. \tag{5}$$

If $t = n-1$, it follows from (5), that $k_1 = 0$, $k_2 = n-1$, and $p_i = 3$ for $1 \leqslant i \leqslant t$. □

3. $f(5) = 4$

It is trivial that $f(3) = 1$. From (1) it follows $f(4) \leqslant 4$, and Lemma 1 gives examples of 4 points saturating 4 unit circles, so that $f(4) = 4$. (A unit circle is called *saturated* if at least three of the n points lie on this unit circle.)

To prove $f(5) \leqslant 4$ three cases are considered.

Case 1. At most 2 saturated unit circles pass through each of the 5 points. Then there are at most $\frac{1}{3} \cdot 2 \cdot 5 < 4$ unit circles, since each circle is counted at least three times.

Case 2. There is one point, say P_5, which lies on 4 saturated unit circles C_1, C_2, C_3, and C_4. Using Lemma 2, it can be assumed that the pair P_1P_2 lies on C_1, P_1P_3

on C_2, P_2P_4 on C_3, and P_3P_4 on C_4. Now a fifth saturated unit circle has to be among $P_1P_2P_3$, $P_1P_2P_4$, $P_1P_3P_4$, or $P_2P_3P_4$. Then by Lemma 1 also $P_2P_3P_5$, $P_1P_4P_5$, $P_1P_4P_5$, or $P_2P_3P_5$ determines a unit circle. In all four cases 4 points saturate 4 circles, and thus the fifth point cannot determine a further saturated circle.

Case 3. One point, say P_5, lies on exactly three saturated unit circles C_1, C_2, and C_3. At least 2 of their further intersections belong to the 5 points, otherwise at least one circle through P_5 passes only through 2 points. Thus C_2 and C_3 may intersect C_1 in P_1 and P_2. Avoiding 4 points saturating 4 circles the points P_3 and P_4 have to lie elsewhere on C_2 and C_3, not on the 2 unit circles through P_1 and P_2. Thus the only possibilities for C_4 and C_5 are $P_1P_3P_4$, and $P_2P_3P_4$. However, the rhombs $P_5M_3P_2M_1$, $M_3P_4M_5P_2$, $P_4M_4P_3M_5$, $M_4P_1M_2P_3$, and $P_1M_1P_5M_2$, where M_i denotes the center of C_i, yield in this sequence the equality of the unit vectors from P_5 to M_1, M_3 to P_2, P_4 to M_5, M_4 to P_3, P_1 to M_2, and M_1 to P_5, which is a contradiction.

Together with $f(5) \geqslant f(4) = 4$ thus $f(5) = 4$ is proved.

4. $f(6) = 8$ and $f(7) = 12$

From (3) with $x = 4$ it follows $f(6) \geqslant 8$. Since any of the 6 partial sets with 5 of the 6 points saturates at most $f(5) = 4$ unit circles, and each circle is counted in 3 partial sets, it follows $f(6) \leqslant \frac{1}{3} \cdot 6 \cdot 4 = 8$, so that $f(6) = 8$.

To show $f(7) \leqslant 12$, we assume there are more than 12 unit circles C_i saturated by 7 points P_j. If 4 of the 7 points saturate 4 circles, then there are at most 10 saturated unit circles since at most each pair of the 3 remaining points can determine at most 2 further saturated unit circles.

If at most 3 points P_j each lie on 6 circles C_i, then there are at most $\frac{1}{3}(3 \cdot 6 + 4 \cdot 5)$, that means at most 12 circles C_i.

Let P_7 lie on the 6 distinct circles $C_1, \ldots, C_6$. By Lemma 2, and avoiding 4 points P_j saturating 4 circles it can be assumed that the pair P_1P_2 lies on C_1, P_1P_3 on C_2, P_2P_4 on C_3, P_3P_5 on C_4, P_4P_6 on C_5, and P_5P_6 on C_6. The triples $P_1P_2P_3$, $P_1P_2P_4$, and $P_1P_3P_5$ cannot determine unit circles since otherwise by Lemma 1 also $P_2P_3P_7$, $P_2P_4P_7$, and $P_3P_5P_7$ are unit circles. In all three cases 4 points saturate 4 circles. If now P_1 is a second point which lies on 6 circles C_i, then the 4 additional circles C_7, C_8, C_9, C_{10} through P_1 have to be among the 7 triples $P_1P_2P_5$, $P_1P_2P_6$, $P_1P_3P_4$, $P_1P_3P_6$, $P_1P_4P_5$, $P_1P_4P_6$, and $P_1P_5P_6$. There are only three possibilities such that P_1P_2 and P_1P_3 occur at most once, and all other pairs of points occur at most twice:

(I) $P_1P_2P_5$, $P_1P_3P_4$, $P_1P_4P_6$, $P_1P_5P_6$;
(II) $P_1P_2P_5$, $P_1P_3P_6$, $P_1P_4P_6$, $P_1P_4P_5$;
(III) $P_1P_2P_6$, $P_1P_3P_4$, $P_1P_5P_6$, $P_1P_4P_5$.

Let M_{ijk} denote the center of the unit circle through $P_iP_jP_k$, and z_i for $1 \leq i \leq 6$ the unit vector from P_7 to the center of C_i. So for example $P_1 = z_1 + z_2$, or $P_5 = z_4 + z_6$, and $M_{156} = z_4 + z_5 + z_6$, or $M_{125} = z_1 + z_2 + z_3$ by Lemma 1.

In (I) the rhomb $P_1M_{156}P_5M_{125}$ yields $P_1M_{125} = z_3 = -P_5M_{156} = -z_5$, which gives the contradiction $P_4 = P_7$.

In (II) the rhomb $P_1M_{146}P_6M_{136}$ yields

$$P_1M_{136} = z_4 = -P_6M_{146} = -z_3. \tag{6}$$

From

$$\begin{aligned} M_{125} &= z_1 + z_2 + z_3 = z_4 + z_6 + x_1, \\ M_{136} &= z_1 + z_2 + z_4 = z_5 + z_6 + x_2, \end{aligned} \tag{7}$$

with unit vectors $x_1 = P_5M_{125}$, $x_2 = P_6M_{126}$, it follows by subtraction

$$z_3 + z_5 + x_2 = 2z_4 + x_1, \tag{8}$$

and together with (6)

$$3z_3 + z_5 + x_2 = x_1. \tag{9}$$

Since all variables in (9) are unit vectors $z_5 = -z_3$ is forced by (9), and with (6) the contradiction $z_5 = z_4$ is received.

In (III) in a similar way

$$z_4 = -z_3 \tag{10}$$

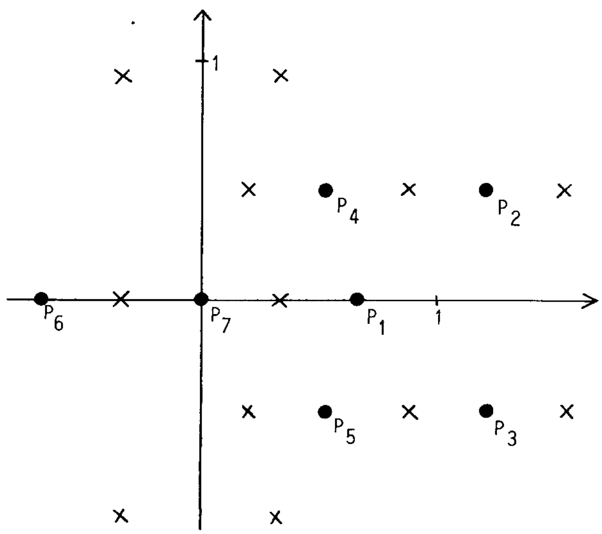

Fig. 1. Seven points which determine 12 saturated unit circles: $P_1(\frac{2}{3}, 0)$, $P_2(\frac{1}{3}(1+\sqrt{7}), \frac{1}{3}\sqrt{2})$, $P_3(\frac{1}{3}(1+\sqrt{7}), -\frac{1}{3}\sqrt{2})$, $P_4(\frac{1}{3}(-1+\sqrt{7}), \frac{1}{3}\sqrt{2})$, $P_5(\frac{1}{3}(-1+\sqrt{7}), -\frac{1}{3}\sqrt{2})$, $P_6(-\frac{2}{3}, 0)$, $P_7(0, 0)$.

is deduced from the rhomb $P_1M_{156}P_6M_{126}$. Then

$$\begin{aligned} M_{126} &= z_1 + z_2 + z_3 = z_5 + z_6 + x_3, \\ M_{134} &= z_1 + z_2 + z_4 = z_3 + z_5 + x_4, \end{aligned} \tag{11}$$

with unit vectors $x_3 = P_6M_{126}$, $x_4 = P_4M_{134}$, yield

$$2z_3 + x_4 = z_4 + z_6 + x_3, \tag{12}$$

and together with (10)

$$3z_3 + x_4 = z_6 + x_3, \tag{13}$$

which forces the contradiction $z_3 = z_6$.

So never more than 12 saturated unit circles C_i are possible, and $f(7) \leqslant 12$ is proved.

It was the more tedious part of the proof of $f(7) = 12$ to find the 7 points of Fig. 1, which shows $f(7) \geqslant 12$. Up to symmetry there exists no other configuration of 7 points which determine 12 saturated unit circles.

If $P_8(\frac{2}{3}\sqrt{7}, \frac{2}{3}\sqrt{2})$ is added in Fig. 1, then P_8 together with P_2P_3, P_2P_4, and P_4P_5 gives 3 further ssturated circles which implies $f(8) \geqslant 15$. It can be conjectured $f(8) = 16$.

Note added in proof

In the meantime $f(8) = 16$ was proved (see [3]).

References

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